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A NEW GENERALIZATION OF C_{pm} FOR PROCESSES WITH ASYMMETRIC TOLERANCES

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Greenwich and Jahr-Schaffrath[9] introduced the incapability index C_{pp}, which is a simple transformation of the index C^*_{pm} proposed by Chan *et al.*[3] Chen[10] considered the incapability index C''_{pp}, a generalization of C_{pp}, to handle processes with asymmetric tolerances. Based on the same idea on C''_{pp}, we consider a new generalization C''_{pm}, which is a modification of the process capability index C_{pm}. In the cases of symmetric tolerances, the new generalization C''_{pm} reduces to the original index C_{pm}. The new generalization C''_{pm} not only takes the proximity of the target value into consideration, like those of C_{pm} and C^*_{pm}, but also takes into account the asymmetry of the specification limits. We compare the new generalization C''_{pm} with $C_{\mathrm{pa}}(1,3)$ and $C_{\mathrm{pa}}(0,4)$, two special cases of $C_{\mathrm{pa}}(u,v)$ recommended by Vännman[7] for asymmetric tolerances. We also investigate the statistical properties of the natural estimator $\hat{C}''_{\mathrm{pm}}$, assuming the process is normally distributed. We obtain the exact distribution and an explicit form of the probability density function of $\hat{C}''_{\mathrm{pm}}$. In addition, we compute the rth moment-expected value, variance of $\hat{C}''_{\mathrm{pm}}$, and analyze the bias as well as the MSE of $\hat{C}''_{\mathrm{pm}}$.

Keywords: Process Capability Index; Target Value; Process Targeting; Asymmetric Tolerances.

1. Introduction

Process capability indices (PCIs), whose purpose is to provide a numerical measure on whether a production process is capable of producing items satisfying the quality requirement preset in the factory, have received substantial research attention. Kane[1] considered two basic indices C_{p} and C_{pk}, and investigated some properties of their estimators. Boyles[2] noted that C_{p} and C_{pk} are yield-based indices. In fact, the designs of C_{p} and C_{pk} are independent of the target value T, which can fail to account for process targeting (the ability to cluster around the target). For this reason, Chan *et al.*[3] developed the index C_{pm}, which takes the process targeting into consideration. We note that the index $C_{\rm pm}$ is not originally designed to provide an exact measure on the number of nonconforming items. But $C_{\rm pm}$ includes the process departure $(\mu - T)^2$ (rather than 6σ alone) in the denominator of the definition to reflect the degree of process targeting. The index $C_{\rm pm}$ is defined as the following:

$$C_{\rm pm} = \frac{\rm USL - LSL}{6\sqrt{\sigma^2 + (\mu - T)^2}}, \tag{1}$$

where USL is the upper specification limit, LSL is the lower specification limit, μ is the process mean, σ is the process standard deviation, and T is the target value.

Recently, many widely used statistical packages and quality researchers addressed process capability applying $C_{\rm pm}$ for cases in which the specification tolerances are asymmetric (see, e.g., Refs. 4 and 5). Boyles[6] noted that such applications can either understate or overstate the process capability in many cases (depending on the position of μ relative to T). A simple generalization of $C_{\rm pm}$ was proposed to handle processes with asymmetric tolerances. The generalization shifts one of the two specification limits, so that the new (shifted) specification limits are symmetric to the target value T (see Kane[1] and Chan *et al.*[3]). The generalization may be defined as the following:

$$C^*_{\rm pm} = \frac{d^*}{3\sqrt{\sigma^2 + (\mu - T)^2}}, \tag{2}$$

where $d^* = \min\{D_l, D_u\}$, $D_u = {\rm USL} - T$, and $D_l = T - {\rm LSL}$. Obviously, if $D_u = D_l$, then $T = m = ({\rm USL} + {\rm LSL})/2$ and $d^* = d = ({\rm USL} - {\rm LSL})/2$; the specification tolerance becomes symmetric and the generalization $C^*_{\rm pm}$ defined in Eq. (2) reduces to the original index $C_{\rm pm}$ defined in Eq. (1).

We note that this generalization can understate process capability by restricting the process to a proper subset of the actual specification range, as noted by Boyles.[6] For processes E and F with $\sigma_{\rm E} = \sigma_{\rm F}$, $\mu_{\rm E} < T, \mu_{\rm F} > T$, satisfying the relationship $|\mu_{\rm F} - T| = |T - \mu_{\rm E}|$ (equal absolute departure), the index values given to processes E and F are the same. For example, consider the following two processes E and F with target value $T = \{3({\rm USL}) + ({\rm LSL})\}/4$, $\mu_{\rm E} = T - 0.5d = m$, $\mu_{\rm F} = T + 0.5d = {\rm USL}$, and $\sigma_{\rm E} = \sigma_{\rm F} = d/6$. For the two processes E and F, we have $|\mu_{\rm F} - T| = |T - \mu_{\rm E}| = 0.5d$ and the same $C^*_{\rm pm} = 0.316$. But, process E is significantly better than process F, as the expected proportions of nonconforming items are approximately 0% and 50% for processes E and F, respectively. Therefore, the index $C^*_{\rm pm}$ inconsistently measures process capability in this case.

Vännman[7,8] investigated a general class of capability indices for processes with asymmetric tolerances. Vännman's generalizations have been defined as

$$C_{\rm pa}(u, v) = \frac{d - |\mu - m| - u|\mu - T|}{3\sqrt{\sigma^2 + v(\mu - T)^2}},$$

where $u, v \geq 0$. Vännman[7] showed that among many (u, v) values, $(u, v) = (1, 3)$ and $(u, v) = (0, 4)$ generate two indices which are most sensitive to process departure from the target value. For $u \geq 1$, the indices $C_{\rm pa}(u,v)$ decrease when mean μ shifts away from target T in either direction. In fact, $C_{\rm pa}(u,v)$ decrease faster when μ shifts away from T to the closer specification limit than that to the farther specification limit. This is an advantage since the index would respond faster to the shift towards "the wrong side" of T than towards the middle of the specification interval (see Ref. 7).

2. A New Generalization $C''_{\rm pm}$

Greenwich and Jahr-Schaffrath[9] introduced the incapability index $C_{\rm pp} = (1/C^*_{\rm pm})^2$, a simple transformation of $C^*_{\rm pm}$, which provides an uncontaminated separation between information concerning the process accuracy and precision while such separated information is not available with the index $C^*_{\rm pm}$. Since $C_{\rm pp}$ inherited the designs (hence the shortcomings) of $C^*_{\rm pm}$, $C_{\rm pm}$ does not reflect process incapability accurately for processes with asymmetric tolerances. Chen[10] considered a generalization of $C_{\rm pp}$ defined as $C''_{\rm pp} = (A/D)^2 + (\sigma/D)^2$, where $D = d^*/3$ and $A = \max\{d(\mu - T)/D_u, d(T-\mu)/D_l\}$, to handle processes with asymmetric tolerances. The generalization $C''_{\rm pp}$ incorporates the asymmetry of the specification tolerances D_u and D_l, and hence reflects process capability more accurate than $C_{\rm pp}$. Both indices $C_{\rm pp}$ and $C''_{\rm pp}$ measure process incapability, which assume a smaller value for a more capable process. Based on the same idea on $C''_{\rm pp}$, we can consider the following generalization of $C_{\rm pm}$ for asymmetric tolerances. The new generalization $C''_{\rm pm}$, which assumes a larger value for a more capable process (same as those of traditional indices $C_{\rm p}$, $C_{\rm pk}$, $C_{\rm pm}$ and $C^*_{\rm pm}$), can be defined as

$$C''_{\rm pm} = \frac{d^*}{3\sqrt{\sigma^2 + A^2}}\,. \tag{3}$$

Obviously, if $T = m = (\mathrm{USL} + \mathrm{LSL})/2$ (symmetric tolerance), then $d = d^*$, $A = |\mu - T|$, and the generalization $C''_{\rm pm}$ reduces to the original index $C_{\rm pm}$. We note that $C''_{\rm pm} \geq 0$ for a process with mean falling within the tolerance limits, which is same as those of the index $C_{\rm pa}(0,4)$ and the yield-based index $C_{\rm pk}$. However, according to today's modern quality improvement theories, reduction of variation from the target is as important as meeting the specifications. The factor A in the definition ensures that the generalization $C''_{\rm pm}$ obtains its maximal value at $\mu = T$ (process is on target) regardless of whether the tolerances are symmetric ($T = m$) or asymmetric ($T \neq m$). Further, for processes E and F with $\sigma_{\rm E} = \sigma_{\rm F}$, $\mu_{\rm E} < T$, $\mu_{\rm F} > T$, satisfying the relationship $(\mu_{\rm F} - T)/D_u = (T - \mu_{\rm E})/D_l$ (equal departure ratio), the index values given to processes E and F are the same. In fact, the value of $C''_{\rm pm}$ decreases faster when μ shifts away from T to the closer specification limit than that to the farther specification limit. We note that $C_{\rm pa}(1,3)$ and $C_{\rm pa}(0,4)$ also differentiate those changes. In particular, for processes E and F with $\sigma_{\rm E} = \sigma_{\rm F}$, $\mu_{\rm E} = \mathrm{LSL}$, and $\mu_{\rm F} = \mathrm{USL}$, the index values of $C''_{\rm pm}$ given to processes E and F are the same, which is the same as those of $C_{\rm pa}(0,4)$ and the yield-based index $C_{\rm pk}$.

Further, for normal processes with μ_E = LSL and μ_F = USL, same index values indicate same expected proportions of nonconforming items in this particular case. Further, given $C''_{pm} > c$, we can obtain a bound on $|\mu - T|$,

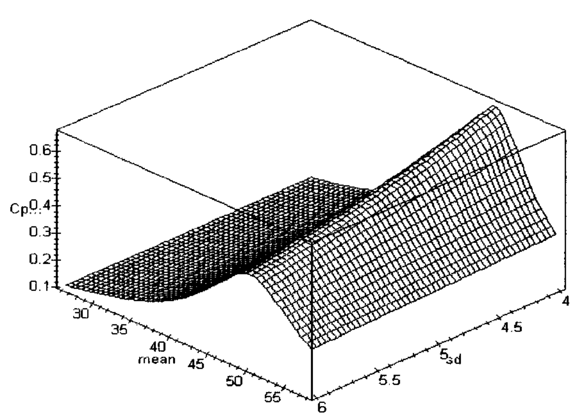
$$T - \{(1 - R)/3c\}D_l < \mu < T + \{(1 - R)/3c\}D_u,$$

where $R = |1 - r|/(1 + r)$, and $r = D_l/D_u$.

We note that the indices C_{pm}, C^*_{pm}, $C_{pa}(1, 3)$, and C''_{pm} obtain their maximal values when the process is on target ($\mu = T$). On the other hand, $C_{pa}(0, 4)$ obtains its maximal value when μ is between T and m, or when $\mu = T = m$.

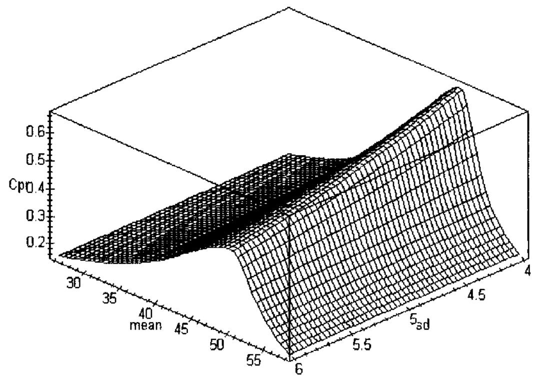
Figures 1(a), 1(b), 1(c) and 1(d) display the surface plots of C^*_{pm}, C''_{pm}, $C_{pa}(1, 3)$, and $C_{pa}(0, 4)$, respectively, for (LSL, T, USL) = (26, 50, 58), $26 \leq \mu \leq 58$ and $4 \leq \sigma \leq 6$. Figures 2(a) and 2(b) display the plots of C^*_{pm} and C''_{pm}; Figs. 3(a) and 3(b) display the plots of $C_{pa}(1, 3)$ and C''_{pm}; Figs. 4(a) and 4(b) display the plots of $C_{pa}(0, 4)$ and C''_{pm} for (LSL, T, USL) = (26, 50, 58) and $26 \leq \mu \leq 58$, with $\sigma = 8/3$ and $\sigma = 16/3$, respectively.

We conclude this section with some comparisons of the proposed index C''_{pm} with the other indices. First, as we pointed out earlier, C^*_{pm} cannot differentiate processes capabilities accurately because the index values of C^*_{pm} given to two processes with equal variance and equal absolute departure are the same. The measure is not reasonable, particularly for processes with asymmetric tolerances. On the other hand, the index values of C''_{pm} given to processes with equal variance and equal departure ratio are the same. We note that both processes have equal average loss in this case. Therefore, we may conclude that C''_{pm} is better than

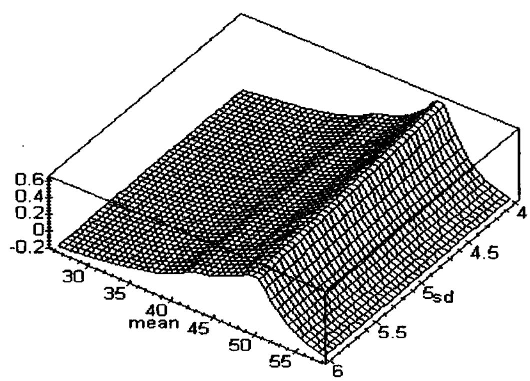


(a)

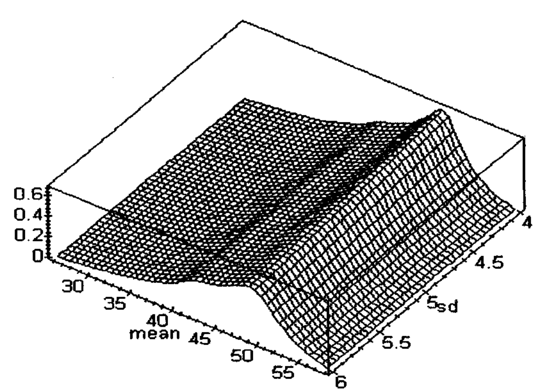
Fig. 1. Surface plot for (LSL, T, USL) = (26, 50, 58), $26 \leq \mu \leq 58$ and $4 \leq \sigma \leq 6$ of (a) C^*_{pm} and (b) C''_{pm}, (c) $C_{pa}(1, 3)$, and (d) $C_{pa}(0, 4)$.



(b)

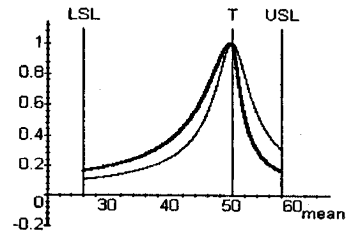


(c)

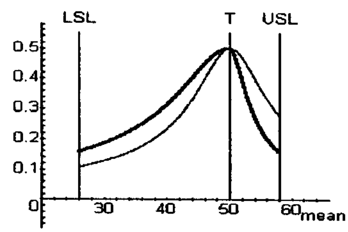


(d)

Fig. 1. (*Continued*)

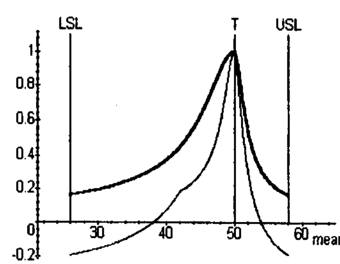


(a)

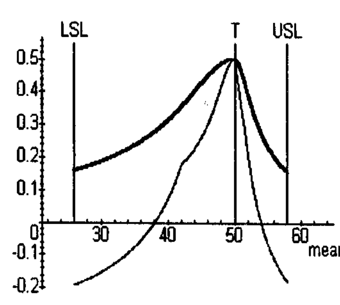


(b)

Fig. 2. Plot of C^{*}_{pm} (thin) and C''_{pm} (bold) for (a) (LSL, T, USL) = (26, 50, 58), $26 \leq \mu \leq 58$ and $\sigma = 8/3$ and (b) (LSL, T, USL) = (26, 50, 58), $26 \leq \mu \leq 58$ and $\sigma = 16/3$.



(a)



(b)

Fig. 3. Plot of $C_{pa}(1, 3)$ (thin) and C''_{pm} (bold) for (a) (LSL, T, USL) = (26, 50, 58), $26 \leq \mu \leq 58$ and $\sigma = 8/3$ and (b) (LSL, T, USL) = (26, 50, 58), $26 \leq \mu \leq 58$ and $\sigma = 16/3$.

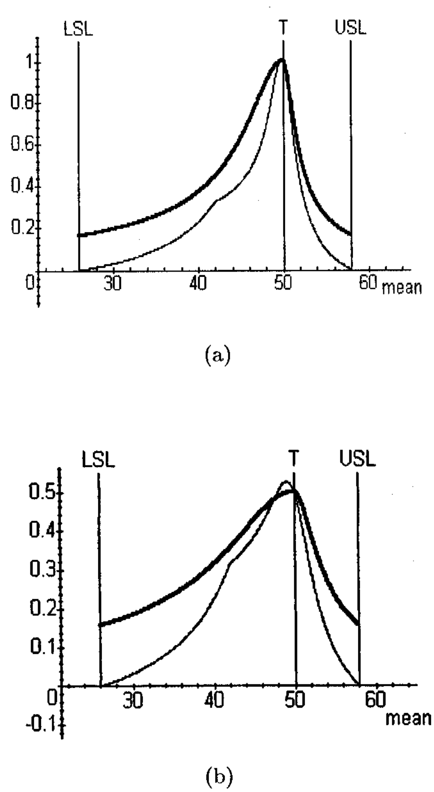


Fig. 4. Plot of $C_{\rm pa}(0,4)$ (thin) and $C''_{\rm pm}$ (bold) for (a) (LSL, T, USL) = (26, 50, 58), $26 \le \mu \le 58$ and $\sigma = 8/3$ and (b) (LSL, T, USL) = (26, 50, 58), $26 \le \mu \le 58$ and $\sigma = 16/3$.

$C^*_{\rm pm}$ based on the criteria of process targeting which is related to process loss. Second, all the indices $C''_{\rm pm}$, $C_{\rm pa}(1,3)$ and $C_{\rm pa}(0,4)$ decrease faster when μ shifts away from T to the closer specification limit than that to the farther specification limit. Third, both $C''_{\rm pm}$ and $C_{\rm pa}(1,3)$ obtain their maximal values when the process is on target ($\mu = T$). Finally, both $C''_{\rm pm}$ and $C_{\rm pa}(0,4)$ are no less than zero for a process with process mean μ falling within the tolerance limits, we note that yield-based index $C_{\rm pk}$ also has the property.

3. Estimation of $C''_{\rm pm}$ and the Sampling Distribution

Let $X_1, X_2, \ldots, X_n$ be a random sample taken from a normal distribution $N(\mu, \sigma^2)$ with mean μ and variance σ^2. To estimate the new generalization $C''_{\rm pm}$, we consider the natural estimator defined as the following:

$$\hat{C}''_{\rm pm} = \frac{d^*}{3\sqrt{S_n^2 + \hat{A}^2}}, \tag{4}$$

where $\hat{A} = \max\{d(\bar{X} - T)/D_u, d(T - \bar{X})/D_l\}$, $\bar{X} = (\sum_{i=1}^n X_i)/n$ and $S_n^2 = \{\sum_{i=1}^n (X_i - \bar{X})^2\}/n$, which may be obtained from a process that is demonstrably stable (in control). If the production tolerance is symmetric, then $\hat{A}$ may be simplified as $|\bar{X}-T|$ and the estimator $\hat{C}''_{\rm pm}$ becomes $\hat{C}_{\rm pm}=d^*/\{3[\sum_{i=1}^{n}(X_i-T)^2/n]^{1/2}\}$, the natural estimator of $C_{\rm pm}$ discussed by Boyles.[2] We note that the natural estimator $\hat{C}''_{\rm pm}$ can be rewritten as

$$\hat{C}''_{\rm pm}=\frac{C}{3\sqrt{K+Y}}, \tag{5}$$

where $C=n^{1/2}d^*/\sigma$, $K=nS_n^2/\sigma^2$, and $Y=[\max\{(d/D_u)Z,-(d/D_l)Z\}]^2$ with $Z=n^{1/2}(\bar{X}-T)/\sigma$. On the assumption of normality, the statistic K is distributed as χ^2_{n-1}, Z is distributed as $N(\delta,1)$, $\delta=n^{1/2}(\mu-T)/\sigma$, and the probability density function of Y is

$$f_Y(y)=\frac{1}{2\sqrt{y}}\left(\frac{1}{d_1}f_z(-\sqrt{y}/d_1)+\frac{1}{d_2}f_z(\sqrt{y}/d_2)\right), \qquad y>0, \tag{6}$$

where $d_1=d/D_l$, and $d_2=d/D_u$. Therefore, the probability density function of $\hat{C}''_{\rm pm}$ can be expressed as (see Appendix A)

$$f_{\hat{C}''_{\rm pm}}(x)=\frac{C^3}{27x^4}\int_0^1\frac{1}{\sqrt{t}}f_k\left(\frac{C^2(1-t)}{9x^2}\right)\left\{\frac{1}{d_1}f_z\left(-\frac{C\sqrt{t}}{3xd_1}\right)\right.$$
$$\left.+\frac{1}{d_2}f_z\left(\frac{C\sqrt{t}}{3xd_2}\right)\right\}dt, \qquad x>0. \tag{7}$$

We note that the statistic Z^2 follows a noncentral chi-square distribution with one degree of freedom and noncentrality parameter δ^2. Chen[10] defined the distribution of Y as a *weighted* noncentral chi-square distribution with one degree of freedom and noncentrality parameter δ^2, under the assumption of normality. Chen[10] also derived the probability density function of Y, in an alternative form of Eq. (6), as

$$f_Y(y)=\frac{\exp(-\frac{\lambda}{2})}{2\sqrt{\pi}}\sum_{j=0}^{\infty}\left\{\frac{(\sqrt{2}\delta)^j}{j!}\Gamma\left(\frac{1+j}{2}\right)\sum_{i=1}^{2}\frac{(-1)^{ij}}{d_i^2}f_{Y_j}(y/d_i^2)\right\}, \qquad y>0, \tag{6'}$$

where $\lambda=\delta^2$ and Y_j is distributed as χ^2_{1+j}. Therefore, the probability density function of $\hat{C}''_{\rm pm}$, in an alternative form of Eq. (7), can be expressed as (see Appendix A)

$$f_{\hat{C}''_{\rm pm}}(x)=\frac{2^{1-n/2}C^n x^{-(n+1)}}{3^n\Gamma((n-1)/2)}\times\frac{\exp(-\frac{\lambda}{2})}{2\sqrt{\pi}}\sum_{j=0}^{\infty}\frac{1}{j!}\left(\frac{\delta C}{3x}\right)^j\sum_{i=1}^{2}(-1)^{ij}$$
$$\times\left(d_i^{-(j+1)}\int_0^1(1-y)^{(n-3)/2}y^{(j-1)2}\exp\left\{\frac{-C^2}{18x^2}(1-y+d_i^{-2}y)\right\}dy\right),$$
$$x>0. \tag{7'}$$

If the production tolerance is symmetric ($d_1 = d_2 = 1$), then the probability density function of Y reduces to the ordinary noncentral chi-square distribution with one degree of freedom and noncentrality parameter $\lambda = \delta^2$, and the probability density function of $\hat{C}''_{\rm pm}$ becomes

$$f_{\hat{C}''_{\rm pm}}(x) = \frac{2^{1-n/2}C^n}{3^n x^{n+1}} \exp\left(-\frac{\lambda}{2} - \frac{C^2}{18x^2}\right) \sum_{j=0}^{\infty} \left\{ \left(\frac{\lambda C^2}{36x^2}\right)^j \Big/ \left(j!\Gamma\left(\frac{n}{2}+j\right)\right) \right\},$$
$$x > 0, \tag{8}$$

which is the probability density function of $\hat{C}_{\rm pm}$ (see Ref. 11).

4. Moments

To obtain the rth moment of $\hat{C}''_{\rm pm}$, we apply the method used in Pearn *et al.*[12] and Vännman[13] for calculating the moment of $\hat{C}_{\rm pm}$. On the assumption of normality, since Y follows a *weighted* noncentral chi-square distribution with one degree of freedom and noncentrality parameter δ^2, and Y is independent with K, then the rth moment of $\hat{C}''_{\rm pm}$ is (see Appendix B)

$$E(\hat{C}''_{\rm pm})^r = \left(\frac{C}{3\sqrt{2}}\right)^r \frac{\exp(-\frac{\lambda}{2})}{2\sqrt{\pi}}$$
$$\times \sum_{j=0}^{\infty} \left\{ \frac{(\sqrt{2}\delta)^j}{j!} \Gamma\left(\frac{1+j}{2}\right) \Gamma\left(\frac{n-r+j}{2}\right) \Big/ \Gamma\left(\frac{n+j}{2}\right) \right\}$$
$$\times \{(-1)^j {}_2F_1(a,b;c;z_1) + {}_2F_1(a,b;c;z_2)\}, \tag{9}$$

where ${}_2F_1(a,b;c;z_i)$ is the Gaussian hypergeometric function (see, e.g., Ref. 14) with parameters $a = r/2$, $b = (1+j)/2$, $c = (n+j)/2$, and $z_i = 1 - d_i^2 (i = 1, 2)$. In particular,

$$E(\hat{C}''_{\rm pm}) = \frac{C \exp\left(-\frac{\lambda}{2}\right)}{6\sqrt{2\pi}} \sum_{j=0}^{\infty} \left\{ \frac{(\sqrt{2}\delta)^j}{j!} \Gamma\left(\frac{1+j}{2}\right) \Gamma\left(\frac{n-1+j}{2}\right) \Big/ \Gamma\left(\frac{n+j}{2}\right) \right\}$$
$$\times \{(-1)^j {}_2F_1(1/2,b;c;z_1) + {}_2F_1(1/2,b;c;z_2)\}, \tag{10}$$

$$E(\hat{C}''_{\rm pm})^2 = \frac{C^2 \exp\left(-\frac{\lambda}{2}\right)}{18\sqrt{\pi}} \sum_{j=0}^{\infty} \left\{ \frac{(\sqrt{2}\delta)^j}{j!} \Gamma\left(\frac{1+j}{2}\right) /(n-2+j) \right\}$$
$$\times \{(-1)^j {}_2F_1(1,b;c;z_1) + {}_2F_1(1,b;c;z_2)\}, \tag{11}$$

$$\mathrm{Var}(\hat{C}''_{\rm pm}) = E(\hat{C}''_{\rm pm})^2 - E^2(\hat{C}''_{\rm pm}). \tag{12}$$

If the production tolerance is symmetric, then $z_i = 0$, ${}_2F_1(1, b; c; 0) = 0$, and

$$E(\hat{C}''_{\rm pm})^r = \left(\frac{C}{3\sqrt{2}}\right)^r \sum_{j=0}^{\infty} \left\{ \frac{\exp(-\frac{\lambda}{2})(\lambda/2)^j}{j!} \Gamma\left(\frac{n-r}{2} + j\right) \Big/ \Gamma\left(\frac{n}{2} + j\right) \right\}, \tag{13}$$

which is the rth moment of $\hat{C}_{\rm pm}$ (see Ref. 13).

We note that the estimator $\hat{C}''_{\rm pm}$ is biased. The magnitude of the bias is $B(\hat{C}''_{\rm pm}) = E(\hat{C}''_{\rm pm}) - C''_{\rm pm}$. The mean square error can be expressed as $\text{MSE}(\hat{C}''_{\rm pm}) = \text{Var}(\hat{C}''_{\rm pm}) + B^2(\hat{C}''_{\rm pm})$. To investigate the behavior of the estimator $\hat{C}''_{\rm pm}$, the bias and the mean square error are calculated (using Maple V computer software) for various values of $\xi = (\mu - T)/\sigma$, d^*/σ, $d_1 = d/D_l$, $d_2 = d/D_u$, and sample size n. Tables 1, 2 and 3 display the values of $B(\hat{C}''_{\rm pm})$ and $\text{MSE}(\hat{C}''_{\rm pm})$ for $\xi = -1.0(0.5)1.0$, $d_1 = 5/6$, $d_2 = 5/4$, and $n = 10(10)50$, with $d^*/\sigma = 3$, 4 and 5, respectively.

The results in Tables 1, 2 and 3 indicate that as $|\xi|$ increases, the mean square error decreases. Further, as the sample size n increases, both the bias and the mean square error decrease. Figure 5 displays the plot of the bias of $\hat{C}''_{\rm pm}$ (versus n) with $\xi = 1.0$, -1.0 and 0 (from bottom to top in the plot) for fixed

Table 1. The values of $B(\hat{C}''_{\rm pm})$ and $\text{MSE}(\hat{C}''_{\rm pm})$ for $d^*/\sigma = 3$, $\xi = -1.0(0.5)1.0$, $d_1 = 5/6$, $d_2 = 5/4$, and $n = 10(10)50$.

n	$\xi = -1.0$		$\xi = -0.5$		$\xi = 0$		$\xi = 0.5$		$\xi = 1.0$	
	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
10	0.0518	0.0321	0.0828	0.0661	0.0787	0.0824	0.0619	0.0631	0.0393	0.0270
20	0.0244	0.0126	0.0394	0.0252	0.0367	0.0320	0.0291	0.0252	0.0185	0.0108
30	0.0160	0.0078	0.0258	0.0153	0.0239	0.0196	0.0191	0.0157	0.0121	0.0067
40	0.0119	0.0056	0.0191	0.0110	0.0178	0.0141	0.0142	0.0114	0.0090	0.0048
50	0.0095	0.0044	0.0152	0.0086	0.0141	0.0110	0.0113	0.0089	0.0072	0.0038

Table 2. The values of $B(\hat{C}''_{\rm pm})$ and $\text{MSE}(\hat{C}''_{\rm pm})$ for $d^*/\sigma = 4$, $\xi = -1.0(0.5)1.0$, $d_1 = 5/6$, $d_2 = 5/4$, and $n = 10(10)50$.

n	$\xi = -1.0$		$\xi = -0.5$		$\xi = 0$		$\xi = 0.5$		$\xi = 1.0$	
	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
10	0.0690	0.0571	0.1104	0.1175	0.1049	0.1465	0.0825	0.1122	0.0524	0.0480
20	0.0326	0.0225	0.0525	0.0448	0.0490	0.0568	0.0389	0.0448	0.0247	0.0192
30	0.0213	0.0139	0.0344	0.0273	0.0319	0.0349	0.0255	0.0279	0.0162	0.0119
40	0.0159	0.0100	0.0255	0.0196	0.0237	0.0251	0.0190	0.0202	0.0120	0.0086
50	0.0126	0.0078	0.0203	0.0152	0.0188	0.0196	0.0151	0.0159	0.0096	0.0068

Table 3. The values of $B(\hat{C}''_{\rm pm})$ and $\mathrm{MSE}(\hat{C}''_{\rm pm})$ for $d^*/\sigma = 5$, $\xi = -1.0(0.5)1.0$, $d_1 = 5/6$, $d_2 = 5/4$, and $n = 10(10)50$.

n	$\xi = -1.0$		$\xi = -0.5$		$\xi = 0$		$\xi = 0.5$		$\xi = 1.0$	
	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
10	0.0863	0.0892	0.1380	0.1836	0.1311	0.2289	0.1032	0.1753	0.0655	0.0751
20	0.0407	0.0351	0.0656	0.0700	0.0612	0.0888	0.0486	0.0700	0.0309	0.0300
30	0.0267	0.0217	0.0429	0.0426	0.0399	0.0545	0.0319	0.0436	0.0202	0.0186
40	0.0198	0.0157	0.0319	0.0306	0.0296	0.0392	0.0237	0.0316	0.0150	0.0135
50	0.0158	0.0123	0.0254	0.0238	0.0235	0.0306	0.0189	0.0248	0.0120	0.0105

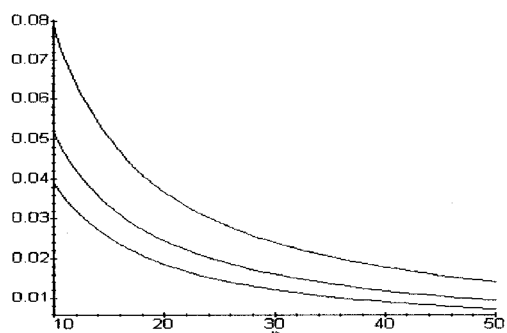


Fig. 5. Bias plot of $\hat{C}''_{\rm pm}$ (versus n) for $d^*/\sigma = 3$, $d_1 = 5/6$, $d_2 = 5/4$ with $\xi = 1.0$, -1.0, and 0 (from bottom to top in the plot).

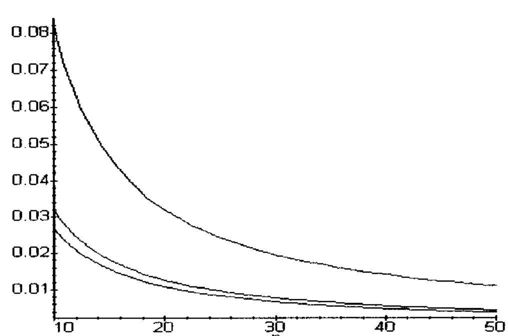


Fig. 6. MSE plot of $\hat{C}''_{\rm pm}$ (versus n) for $d^*/\sigma = 3$, $d_1 = 5/6$, $d_2 = 5/4$ with $\xi = 1.0$, -1.0 and 0 (from bottom to top in the plot).

$d^*/\sigma = 3$, $d_1 = 5/6$, and $d_2 = 5/4$. Figure 6 displays the plot of the MSE of $\hat{C}''_{\rm pm}$ (versus n) with $\xi = 1.0$, -1.0 and 0 (from bottom to top in the plot) for fixed $d^*/\sigma = 3$, $d_1 = 5/6$, $d_2 = 5/4$. Combining the results in Tables 1, 2 and 3, we observe as the value of d^*/σ increases, both the bias and the mean square error also increase for fixed d_1, d_2, ξ and n. Figure 7 displays the plot of the bias of $\hat{C}''_{\rm pm}$

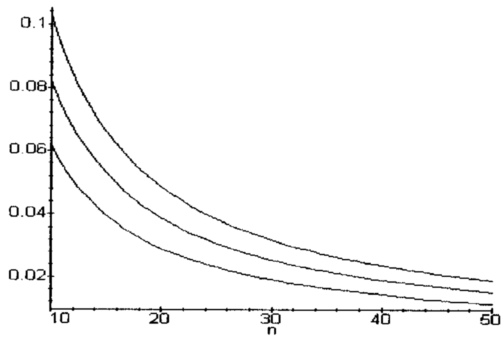


Fig. 7. Bias plot of $\hat{C}''_{\mathrm{pm}}$ (versus n) for $\xi = 0.5$, $d_1 = 5/6$, $d_2 = 5/4$ with $d^*/\sigma = 3$, 4 and 5 (from bottom to top in the plot).

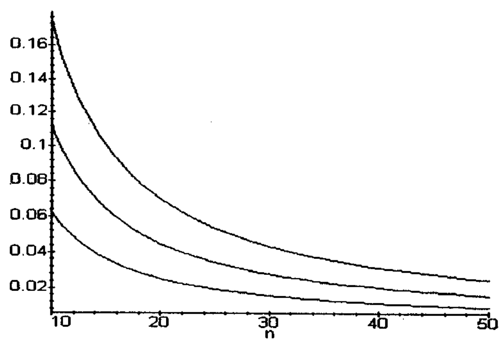


Fig. 8. MSE plot of $\hat{C}''_{\mathrm{pm}}$ (versus n) for $\xi = 0.5$, $d_1 = 5/6$, $d_2 = 5/4$ with $d^*/\sigma = 3$, 4 and 5 (from bottom to top in the plot).

(versus n) with $d^*/\sigma = 3$, 4 and 5 (from bottom to top in the plot) for fixed $\xi = 0.5$, $d_1 = 5/6$ and $d_2 = 5/4$. Figure 8 displays the plot of the MSE of $\hat{C}''_{\mathrm{pm}}$ (versus n) with $d^*/\sigma = 3$, 4 and 5 (from bottom to top in the plot) for fixed $\xi = 0.5$, $d_1 = 5/6$ and $d_2 = 5/4$.

5. Conclusions

In this paper, we considered a new generalization C''_{pm}, a modification of the process capability index C_{pm}, to handle processes with asymmetric tolerances. The new generalization C''_{pm} not only takes the proximity of the target value into consideration, like those of C_{pm} and C^*_{pm}, but also takes into account the asymmetry of the specification limits. We compared C''_{pm} with $C_{\mathrm{pa}}(1,3)$ and $C_{\mathrm{pa}}(0,4)$, two special cases of $C_{\mathrm{pa}}(u,v)$ recommended by Vännman[7] for asymmetric tolerances. The results show that both C''_{pm} and $C_{\mathrm{pa}}(u,v)$ decrease faster when μ shifts away from T to the closer specification limit than that to the farther specification limit. In particular, both C''_{pm} and $C_{\mathrm{pa}}(1,3)$ obtain their maximal values when the process is on target ($\mu = T$). Furthermore, all values of C''_{pm}, $C_{\mathrm{pa}}(0,4)$, and the yield-based index $C_{\rm pk}$ are no less than zero for a process with mean μ falling within the tolerance limits. We also investigated the statistical properties of the natural estimator of $C''_{\rm pm}$ assuming that the process is normally distributed. We obtained the exact distribution, the rth moment, expected value, and the variance of the natural estimator $\hat{C}''_{\rm pm}$. We also analyzed the bias and the MSE. The new generalization $C''_{\rm pm}$ measures process capability more accurately than the original index $C_{\rm pm}$ and other existing generalizations, which can be implemented straightforwardly. Therefore, the new generalization $C''_{\rm pm}$ should be recommended for in-plant applications.

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Appendix A

We note that Y and K are independent under the assumption of normality. Using the representation in Eq. (5) conditioning to Y, we obtain the following cumulative function of $\hat{C}''_{\rm pm}$:

$$\begin{aligned} F_{\hat{C}''_{\rm pm}}(x) &= 1 - \mathrm{P}\left\{\frac{C}{3\sqrt{K+Y}} > x\right\} \\ &= 1 - \int_0^\infty \mathrm{P}\left\{\sqrt{K+Y} < \frac{C}{3x} | Y = y\right\} f_Y(y)dy \\ &= 1 - \int_0^{C^2/(9x^2)} \mathrm{P}\{K < C^2/(9x^2) - y\} f_Y(y)dy\,, \qquad x > 0\,. \end{aligned} \tag{A.1}$$

The last equality in (A.1) holds since $\mathrm{P}\{K < C^2/(9x^2) - y\} = 0$, for $y > C^2/(9x^2)$. Hence, we have

$$F_{\hat{C}''_{\rm pm}}(x) = 1 - \int_0^{C^2/(9x^2)} F_k(C^2/(9x^2) - y) f_Y(y)dy\,. \tag{A.2}$$

Using the representation in Eq. (6), we may obtain

$$\begin{aligned} F_{\hat{C}''_{\rm pm}}(x) = 1 - \int_0^{C^2/(9x^2)} & F_k(C^2/(9x^2) - y)\frac{1}{2\sqrt{y}} \\ & \times \left(\frac{1}{d_1} f_z(-\sqrt{y}/d_1) + \frac{1}{d_2} f_z(\sqrt{y}/d_2)\right) dy \end{aligned} \tag{A.3}$$

$$f_{\hat{C}''_{\rm pm}}(x) = \int_0^{C^2/(9x^2)} f_k(C^2/(9x^2) - y)\frac{C^2}{9x^3\sqrt{y}} \times \left(\frac{1}{d_1} f_z(-\sqrt{y}/d_1) + \frac{1}{d_2} f_z(\sqrt{y}/d_2)\right) dy\,. \quad \text{(A.4)}$$

Changing the variable $t = (3x/C)^2 y$ in integral (A.4), we can obtain the pdf of $\hat{C}''_{\rm pm}$, as expressed in Eq. (7).

Using the expression for the pdf of Y in Eq. (6'), we may obtain

$$F_{\hat{C}''_{\rm pm}}(x) = 1 - \frac{\exp(-\frac{\lambda}{2})}{2\sqrt{\pi}} \sum_{j=0}^{\infty} \frac{(\sqrt{2}\delta)^j}{j!}\Gamma\left(\frac{1+j}{2}\right) \sum_{i=1}^{2} (-1)^{ij} \times \int_0^{C^2/(9x^2 d_i^2)} F_k(C^2/(9x^2) - d_i^2 y) f_{Y_j}(y) dy\,, \quad \text{(A.3')}$$

$$f_{\hat{C}''_{\rm pm}}(x) = \frac{\exp(-\frac{\lambda}{2})}{2\sqrt{\pi}} \sum_{j=0}^{\infty} \frac{(\sqrt{2}\delta)^j}{j!}\Gamma\left(\frac{1+j}{2}\right) \sum_{i=1}^{2} (-1)^{ij} \times \int_0^{C^2/(9x^2 d_i^2)} f_k(C^2/(9x^2) - d_i^2 y) 2C^2/(9x^3) f_{Y_j}(y) dy\,. \quad \text{(A.4')}$$

Changing the variable $t = (3xd_i/C)^2 y$ in integral (A.4'), we can obtain the pdf of $\hat{C}''_{\rm pm}$, as expressed in Eq. (7').

Appendix B

The rth moment of $\hat{C}''_{\rm pm}$ can be calculated as

$$\begin{aligned} E(\hat{C}''_{\rm pm})^r &= (C/3)^r E(K+Y)^{-r/2} \\ &= (C/3)^r \frac{\exp(-\frac{\lambda}{2})}{2\sqrt{\pi}} \sum_{j=0}^{\infty} \left\{\frac{(\sqrt{2}\delta)^j}{j!}\Gamma\left(\frac{1+j}{2}\right)\right\} \\ &\quad \times \{(-1)^j E[K + d_1^2 Y_j]^{-r/2} + E[K + d_2^2 Y_j]^{-r/2}\}\,, \end{aligned} \quad \text{(B.1)}$$

where Y_j is distributed as χ^2_{1+j}. Let $e_j = Y_j/(K+Y_j)$ and $W_j = K + Y_j$. Under the assumption of normality e_j and W_j are independent (see Ref. 15 or 13), and e_j is distributed as beta(α, β) distribution with $\alpha = (1+j)/2$, and $\beta = (n-1)/2$. Further, W_j is distributed as χ^2_{n+j}, a chi-square distribution with $(n+j)$ degrees of freedom. Therefore,

$$E(K + vY_j)^{-r/2} = E(W_j)^{-r/2} E[1 + (v-1)e_j]^{-r/2}\,,$$

$$E(W_j)^{-r/2} = 2^{-r/2}\Gamma\left(\frac{n-r+j}{2}\right) \Big/ \Gamma\left(\frac{n+j}{2}\right)\,,$$

$$E[1 + (v-1)e_j]^{-r/2} = {}_2F_1(a, b; c; z)\,,$$

where ${}_2F_1(a, b; c; z)$ is the Gaussian hypergeometric function with parameters $a = r/2$, $b = (1 + j)/2$, $c = (n + j)/2$, and $z = 1 - v$. Combining the results, we can obtain the rth moment of $\hat{C}''_{\rm pm}$ as expressed in Eq. (9).

References

1. V. E. Kane, "Process capability indices," *J. Qual. Technol.* **18** (1986), pp. 41–52.
2. R. A. Boyles, "The Taguchi capability index," *J. Qual. Technol.* **23** (1991), pp. 17–26.
3. L. K. Chan, S. W. Cheng, and F. A. Spiring, "A new measure of process capability: $C_{\rm pm}$," *J. Qual. Technol.* **20** (1988), pp. 162–175.
4. R. H. Kushler and P. Hurley, "Confidence bounds for capability indices," *J. Qual. Technol.* **24** (1992), pp. 188–195.
5. L. A. Franklin and G. Wasserman, "Bootstrap lower confidence limits for capability indices," *J. Qual. Technol.* **24** (1992), pp. 196–210.
6. R. A. Boyles, "Process capability with asymmetric tolerances," *Communications in Statistics — Simulation and Computation* **23** (1994), pp. 615–643.
7. K. Vännman, "A general class of capability indices in the case of asymmetric tolerances," *Communications in Statistics — Theory and Methods* **26** (1994), pp. 2049–2072.
8. K. Vännman, "Capability indices when tolerances are asymmetric," in *Quality Improvement Through Statistical Methods*, eds. B. Abraham (Birkhauser, Boston), pp. 79–97.
9. M. Greenwich and B. L. Jahr-Schaffrath, "A process incapability index," *International Journal of Quality & Reliability Management* **12** (1995), pp. 58–71.
10. K. S. Chen, "Incapability index with asymmetric tolerances," *Statistica Sinica* **8** (1998), pp. 253–262.
11. K. Vännman and S. Kotz, "A superstructure of capability indices — distributional properties and implications," *Scandinavian Journal of Statistics* **22** (1995), pp. 477–491.
12. W. L. Pearn, S. Kotz, and N. L. Johnson, "Distributional and inferential properties of process capability indices," *J. Qual. Technol.* **24** (1992), pp. 216–231.
13. K. Vännman, "A unified approach to capability indices," *Statistica Sinica* **5** (1995), pp. 805–820.
14. M. Abramowitz and I. A. Stegun (eds.), *Handbook of Mathematical Functions* (Dover Publications, New York, 1970).
15. N. L. Johnson, S. Kotz, and N. Balakrishnan, *Distributions in Statistics. Continuous Univariate Distributions-2*, 2nd edition (John Wiley, New York, 1995).

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